LOCALLY DIVERGENT ORBITS ON HILBERT MODULAR SPACES

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Abstract. We describe the closures of locally divergent orbits under the action of tori on Hilbert modular spaces of rank $r \geq 2$. In particular, we prove that if D is a maximal $\mathbb{R}$-split torus acting on a real Hilbert modular space then every locally divergent non-closed orbit is dense for $r > 2$ and its closure is a finite union of tori orbits for $r = 2$. Our results confirm an orbit rigidity conjecture of G.A.Margulis in all cases except for (i) $r = 2$ and, (ii) $r > 2$ and the Hilbert modular space corresponds to a CM-field; in the cases (i) and (ii) our results contradict the conjecture.

As an application, we describe the set of values at integral points of collections of non-proportional, split, binary, quadratic forms over number fields.

1. Introduction

During the last decade the problems of the descriptions of orbit closures for actions of maximal split tori on homogeneous spaces appear to be among the central ones in homogeneous dynamics. The interest in such problems is motivated to a large extent by number theoretic applications. One example about the efficiency of the homogeneous dynamics approach in the number theory is G.A.Margulis' proof of the long-standing Oppenheim conjecture [M1]. In our days this approach looks quite promising regarding the still open Littlewood conjecture [M2, §2]. We refer to [L] and [M3] for an account of results and conjectures on the subject. In the present paper[1], as an application of the main results, we give an explicit description of the set of values at integral points of a collection of non-proportional, split, binary quadratic forms over number fields.

Let us introduce some basic terminology. Let K be a number field, $\mathcal{O}$ its ring of integers and K_i, $1 \leq i \leq r$, all the archimedean completions of K. *Throughout this paper we assume that* $r \geq 2$. Put $G = \prod_{i=1}^{r} G_i$, where $G_i = \mathrm{SL}(2, K_i)$, and let $\Gamma = \mathrm{SL}(2, \mathcal{O})$ be identified with its

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image in G under the diagonal embedding. The subgroup Γ is a non-uniform irreducible lattice in G and by the arithmeticity theorem (cf. [M4], [M5], [S]) all non-uniform irreducible lattices in G arise by this construction up to conjugation and commensurability. The quotient space G/Γ is called *Hilbert modular space of rank* r. Denote by $\pi : G \to G/\Gamma$ the natural projection. Let D_i be the connected component of the identity of the diagonal subgroup of G_i and let $D_{i,\mathbb{R}}$ be the connected component of the identity of the subgroup of *real* matrices in D_i. So, $D_{i,\mathbb{R}} = D_i$ if $K_i = \mathbb{R}$. For every non-empty $I \subset \{1, \cdots, r\}$ we denote $D_I = \prod_{i\in I} D_i$ and $D_{I,\mathbb{R}} = \prod_{i\in I} D_{i,\mathbb{R}}$. When $I = \{1, \cdots, r\}$ we write D and $D_{\mathbb{R}}$ instead of D_I and $D_{I,\mathbb{R}}$, respectively. By a torus (respectively, an $\mathbb{R}$-split torus or, simply, a split torus) in G we mean a subgroup conjugated to a closed connected subgroup of D (respectively, $D_{\mathbb{R}}$). An orbit $D_I\pi(g)$ is called *locally divergent* if $D_i\pi(g)$ is divergent for all $i \in I$. (Recall that if H is a closed non-compact subgroup of G and $x \in G/\Gamma$ then the orbit Hx is divergent if the orbit map $h \mapsto hx$ is proper or, equivalently, if $\{h_n x\}$ leaves compact subsets of G/Γ whenever h_n leaves compact subsets of H.) The orbit $D_{I,\mathbb{R}}\pi(g)$ is locally divergent if and only if the orbit $D_I\pi(g)$ is locally divergent. The description of the divergent D_i-orbits (and, therefore, the divergent $D_{i,\mathbb{R}}$-orbits) follows from the general results of [T1] (see §2.2).

The following conjecture is a special case of a conjecture of G.A.Margulis [M3, Conjecture 1].

Conjecture A(*orbit rigidity*): If $\#I \geq 2$ then every orbit $D_{I,\mathbb{R}}x$, $x \in G/\Gamma$, has *homogeneous closure*, that is, $\overline{D_{I,\mathbb{R}}x} = Fx$, where F is a closed subgroup in G containing $D_{I,\mathbb{R}}$.

Broadly speaking, the general [M3, Conjecture 1] says that the closure of an orbit for the action of an $\mathbb{R}$-split torus T of dimension ≥ 2 on a homogeneous space of finite volume G/Γ is homogeneous itself provided G/Γ does not admit a real rank 1 T-invariant quotient. An immediate corollary from our Theorem 1.1 shows that Margulis conjecture fails for every Hilbert modular space of rank 2 (Corollary 1.3), for instance, it fails when $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and Γ is the diagonal imbedding of $\mathrm{SL}(2,\sqrt{2})$ in G. We apply this result to produce counterexemples to [M3, Conjecture 1] for much larger classes of homogeneous spaces as $\mathrm{SO}(f,\mathbb{R})/\mathrm{SO}(f,\mathbb{Z})$ and $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}), n \geq 4$ (see Corollary 1.4 and §7). For actions of split tori on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ completely different examples of orbits with non-homogeneous closures contradicting [M3, Conjecture 1] have been first constructed by F.Maucourant

[Ma] when $n \geq 6$ and by U.Shapira [Sha] when $n = 3$. The constructions from [Ma] and [Sha] do not apply to the class of Hilbert modular spaces.

It is instructive to note that the split tori action on homogeneous spaces with finite volume is the counterpart of the unipotent subgroups action on such spaces. The latter action is completely understood in most general setting by M.Ratner [Ra]. (See also the earlier papers [DM1], [M1], [Sh].) One of the basic intrinsic differences between the two actions is that the unipotent orbits never diverge. This is a fundamental result of Margulis [M6] which admits important quantitatif versions (cf.[D], [DM2], [KlM]).

We describe in this paper the closures of locally divergent D_I-orbits on the Hilbert modular spaces G/Γ. It turns out that, on one hand, Conjecture A is not valid for the action of two-dimensional tori (Theorem 1.1) and in the case of Hilbert modular spaces corresponding to CM-fields (Theorem 1.8) and, on the other hand, Conjecture A is valid in all remaining cases (Theorem 1.5).

Let us formulate our theorems. The cases $\#I = 2$ and $\#I > 2$ are very different by nature and will be considered separately.

Theorem 1.1. *Let $\#I = 2$ and $D_I\pi(g)$ be a locally divergent orbit on G/Γ. Suppose that the closure $\overline{D_I\pi(g)}$ is not an orbit of a torus. Then*

$$\overline{D_I\pi(g)} = \bigcup_{i=1}^{s} T_i\pi(h_i) \bigcup D_I\pi(g),$$

where $2 \leq s \leq 4$, T_i are tori containing D_I and $T_i\pi(h_i)$ are pairwise different closed non-compact orbits. In particular, if $\#I = 2$ then there are no dense locally divergent D_I-orbits.

The locally divergent orbits $D_I\pi(g)$, $\#I \geq 2$, such that $\overline{D_I\pi(g)}$ is not an orbit of a torus always exist and are explicitly described by Corollary 1.9 below. Moreover, as shown by Proposition 7.1, there are locally divergent orbits for which the boundaries of their closures consist of exactly $s = 4$ different closed orbits.

Theorem 1.1 easily implies that the orbit rigidity conjecture in the case of Hilbert modular spaces is not valid. More precisely, we have the following:

Corollary 1.2. *Let $\#I = 2$ and $T = D_I$* or *$D_{I,\mathbb{R}}$. Suppose that $T\pi(g)$ is a locally divergent orbit such that $\overline{T\pi(g)}$ is not an orbit of a torus. Then the orbit $T\pi(g)$ is a proper open subset of $\overline{T\pi(g)}$. In particular, $\overline{T\pi(g)}$ is not homogeneous.*

The maximal tori action (the so-called Weyl chamber flow) deserves special attention. The next corollary is a particular case of Theorem 1.1:

Corollary 1.3. *Suppose that the Hilbert modular space G/Γ is of rank $r = 2$. Then a locally divergent orbit $D\pi(g)$ is either closed or*

$$\overline{D\pi(g)} \setminus D\pi(g) = \bigcup_{i=1}^{s} D\pi(h_i),$$

where $2 \leq s \leq 4$, and $D\pi(h_i)$ are pairwise different, closed, non-compact orbits.

After the main results of this paper had been reported [T2], appeared the preprint of E.lindenstrauss and U.Shapira [LS] where, using different ideas, the authors prove a somewhat similar to the above corollary result for the action of maximal tori on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$.

The homogeneous space G/Γ in the formulation of Corollary 1.3 can be embedded as a closed subvariety in a number of homogeneous spaces H/Δ where H is a semi-simple Lie group and Δ its irreducible lattice. We use this to obtain more examples of multidimensional tori orbits with non-homogeneous closures. For instance, at the end of Section 4 we will prove:

Corollary 1.4. *Suppose that one of the following holds:*

(a) $H = \mathrm{SO}(f,\mathbb{R})$ *and* $\Delta = \mathrm{SO}(f,\mathbb{Z})$, *where f is a non-degenerate quadratic form with rational coefficients of $n \geq 5$ variables, of $\mathbb{R}$-rank ≥ 2, and of $\mathbb{Q}$-rank ≥ 1;*

(b) $H = \mathrm{SL}(n,\mathbb{R})$, $\Delta = \mathrm{SL}(n,\mathbb{Z})$ *and* $n \geq 4$.

Let T be a maximal $\mathbb{R}$-split torus of H acting on H/Δ by left multiplication and let $\pi_\circ : H \to H/\Delta,\ g \mapsto g\Delta$. Then there exist orbits $T\pi_\circ(g)$ such that

$$\overline{T\pi_\circ(g)} \setminus T\pi_\circ(g) = \bigcup_{i=1}^{4} T\pi_\circ(h_i)$$

where $T\pi_\circ(h_i),\ 1 \leq i \leq 4$, are pairwise different, closed, non-compact orbits.

Recall that if f is a real isotropic quadratic form of $n = 3$ variables then $\mathrm{SO}(f,\mathbb{R})$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. If $n = 4$, $\mathrm{rank}_{\mathbb{Q}} f = 1$ and $\mathrm{rank}_{\mathbb{R}} f = 2$ then $\mathrm{SO}(f,\mathbb{R})$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(f,\mathbb{Z})$ is an irreducible non-uniform lattice in $\mathrm{SO}(f,\mathbb{R})$ (cf.[A, Theorems 5.21 and 5.22]). If, $n \geq 5$, then $\mathrm{SO}(f)$ is a simple group of type $B_{\frac{n-1}{2}}$ if n is odd and of type $D_{\frac{n}{2}}$ if n is even.

The dynamics of the action of D_I on a Hilbert modular space G/Γ differs drastically when $\#I > 2$. In this case the so-called CM-*fields*

play an important role. Recall that a number field K is called CM-field (so named for a close connection to the theory of complex multiplication) if it is a quadratic extension of a totally real number field which is totally imaginary.

Theorem 1.5. *Let $\#I > 2$ and $D_I\pi(g)$ be a locally divergent orbit such that $\overline{D_I\pi(g)}$ is not an orbit of a torus. Assume that K is not a* CM-*field. Then $D_I\pi(g)$ is a dense orbit.*

If K *is* a CM-field then the closure of $D_I\pi(g)$ might not be homogeneous. This is related to a simple observation which we are going to explain now. Denote by $G_{i,\mathbb{R}},\ 1 \leq i \leq r$, the subgroup of real matrices in G_i and put $G_{\mathbb{R}} = \prod_{i=1}^{r} G_{i,\mathbb{R}}$. Clearly, $G_{\mathbb{R}} \supset D_{I,\mathbb{R}}$. Now let K be a CM-field which is a quadratic extension of a totally real number field F and let $\mathcal{O}_F$ be the ring of integers of F. Then $\Gamma_{\mathbb{R}} = \mathrm{SL}(2, \mathcal{O}_F)$ is a lattice in $G_{\mathbb{R}}$ and the orbit $G_{\mathbb{R}}\pi(e)$ is closed and homeomorphic to $G_{\mathbb{R}}/\Gamma_{\mathbb{R}}$. It is standard to prove that this property characterize K as a CM-field, that is, if G/Γ admits a closed $G_{\mathbb{R}}$-orbit then K is a CM-field. It follows from the special case of Theorem 1.5 for totally real fields (Corollary 1.6) that if K is a CM-field of degree > 4, $x \in G_{\mathbb{R}}\pi(e)$ and $D_{I,\mathbb{R}}x$ is a locally divergent orbit whose closure is not an orbit of a torus, then $\overline{D_{I,\mathbb{R}}x} = G_{\mathbb{R}}\pi(e)$. Since D_I is an extension of $D_{I,\mathbb{R}}$ by a compact torus this implies that $\overline{D_I x} = D_I G_{\mathbb{R}}\pi(e)$. It is clear that $\overline{D_I x}$ is not homogeneous which shows that if K is a CM-field the analog of Theorem 1.5 is not valid.

Let us turn to the study of the orbits for the action of the $\mathbb{R}$-split tori $D_{I,\mathbb{R}}$ which is also important from the point of view of Margulis' conjecture.

In the classical case of *real* Hilbert modular spaces in view of Theorem 1.5 we have:

Corollary 1.6. *Let K be a totally real number field of degree $r \geq 3$. Let $\#I > 2$ and $D_I\pi(g)$ be a locally divergent orbit such that $\overline{D_I\pi(g)}$ is not an orbit of a torus. Then $\overline{D_I\pi(g)} = G/\Gamma$.*

In particular, if $D_I = D$ then $D\pi(g)$ is either closed or dense.

In §5 we prove the following generalization of Corollary 1.6:

Corollary 1.7. *With the assumptions of Theorem 1.5, the orbit $D_{I,\mathbb{R}}\pi(g)$ is dense in G/Γ.*

When K is a CM-field we obtain examples of tori orbits contradicting Margulis' conjecture which are *essentially* different from those provided by Theorem 1.1.

Theorem 1.8. *Let K be a* CM*-field and $\#I > 2$. Then there exists a point $x \in G/\Gamma$ with the following properties:*

(i) *The orbit $D_{I,\mathbb{R}}x$ is locally divergent and $\overline{D_{I,\mathbb{R}}x} \neq G/\Gamma$;*
(ii) *There exists an $y \in \overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x$ such that $\overline{D_{I,\mathbb{R}}x} = \overline{D_{I,\mathbb{R}}y}$ and Hy is not closed for any proper subgroup H of G containing $D_{I,\mathbb{R}}$;*
(iii) *$\overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x$ is not contain in a union of countably many closed orbits of proper subgroups of G.*
In particular, $\overline{D_{I,\mathbb{R}}x}$ is not homogeneous.

As a by-product of the proofs of the above theorems we get the following corollary which is known for $D_I = D$ (see Theorem 2.1 below).

Corollary 1.9. *Let $\#I \geq 2$. Then $\mathcal{N}_G(D_I)G_K \subsetneqq \bigcap_{i\in I}\big(\mathcal{N}_G(D_i)G_K\big)$ and $D_I\pi(g)$ is a locally divergent orbit such that $\overline{D_I\pi(g)}$ is not an orbit of a torus if and only*

$$g \in \bigcap_{i\in I}\big(\mathcal{N}_G(D_i)G_K\big) \setminus \mathcal{N}_G(D_I)G_K.$$

The following orbit rigidity conjecture is plausible:

Conjecture B. Let G be a real semisimple algebraic group with no compact factors and let Γ be an irreducible lattice in G. Suppose that $\mathrm{rank}_{\mathbb{R}}G \geq 2$ and that every semisimple subgroup G_0 in G of the same $\mathbb{R}$-rank as G acts minimally on G/Γ (i.e., every G_0-orbit is dense). Then if T is a maximal $\mathbb{R}$-split torus in G and $x \in G/\Gamma$, either

(1) $\overline{Tx} = G/\Gamma$, or
(2) $\overline{Tx} \setminus Tx \subset \bigcup_{i=1}^{n} H_i x_i$, where H_i are proper reductive subgroups of G containing T and the orbits $H_i x_i$ are closed.

We apply our method to study the values of binary quadratic forms at integral points. Denote $A = \prod_{i=1}^{r} K_i$ and $A^* = \prod_{i=1}^{r} K_i^*$. The polynomial ring $A[X,Y]$ is naturally isomorphic to $\prod_{i=1}^{r} K_i[X,Y]$. The natural embeddings of K into K_i induce embeddings of $K[X,Y]$ into $K_i[X,Y]$, $1 \leq i \leq r$, and a diagonal embedding of $K[X,Y]$ into $A[X,Y]$. *In the next theorem $f = (f_i)_{i\in\overline{1,r}} \in A[X,Y]$, where $f_i \in K_i[X,Y]$ are split, non-degenerate, quadratic forms over K* (that is, $f_i = l_{i,1} \cdot l_{i,2}$, where $l_{i,1}$ and $l_{i,2}$ are linearly independent linear forms with coefficients from K). If $(\alpha,\beta) \in \mathcal{O}^2$ then $f(\alpha,\beta)$ is an element in A with its i-th coordinate equal to $f_i(\alpha,\beta)$. It is clear that if f_i are two by two proportional

(equivalently, if there exists a $g \in K[X,Y]$ such that $f_i = c_i \cdot g$, $c_i \in K$, for all i) then $f(\mathcal{O}^2)$ is a discrete subset of A. It follows from [T1, Theorem 1.8] that the opposite is also valid: the discreteness of $f(\mathcal{O}^2)$ in A implies the proportionality of f_i , $1 \leq i \leq r$. In the next theorem we describe the closure of $f(\mathcal{O}^2)$ in A when f_i, $1 \leq i \leq r$, are not proportional.

Theorem 1.10. *With the above notation and assumptions, suppose that f_i are not proportional. Then the following assertions hold:*

(a) *If $r > 2$ and K is not a* CM*-field then $f(\mathcal{O}^2)$ is dense in A;*
(b) *Let $r = 2$. Put $K'_1 = \{f_1(x,y) : (x,y) \in K_1^2 \text{ and } f_2(x,y) = 0\}$ and $K'_2 = \{f_2(x,y) : (x,y) \in K_2^2 \text{ and } f_1(x,y) = 0\}$. Then there exist $2 \leq s \leq 4$ and pairwise nonproportional K-rational quadratic forms $\phi^{(j)} \in K[X,Y]$, $1 \leq j \leq s$, such that*

$$\overline{f(\mathcal{O}^2)} = \bigcup_{j=1}^{s} \phi^{(j)}(\mathcal{O}^2) \bigcup (K'_1 \times \{0\}) \bigcup (\{0\} \times K'_2) \bigcup f(\mathcal{O}^2).$$

So, the set $\overline{f(\mathcal{O}^2)} \cap A^*$ is countable and the set $\overline{f(\mathcal{O}^2)} \bigcap (A \setminus A^*)$ is continuum. Moreover, $K'_i = \mathbb{C}$ if $K_i = \mathbb{C}$ and $K'_i = \mathbb{R}, \mathbb{R}_-$ or *$\mathbb{R}_+$ if $K_i = \mathbb{R}$.*

Let us describe the organisation and the main points of the paper. In §2 we recall some results from our previous paper [T1] and we prove auxiliary results about the structure of the group of units of a number field. The phenomenon which is at the base of the difference between the 2-dimensional tori action (Theorem 1.1) and the higher dimensional tori action (Theorem 1.5) is the simple fact that the projection of the group of units to any archimedean completion K_v^* of K^* is discrete if $r = 2$ and is not discrete if $r > 2$. In §3 we use dynamical type arguments in combination with Minkowski's theorem for the convex body, the structure of the locally divergent orbits [T1] and the Bruhat decomposition for SL_2 in order to describe in a very explicit way the accumulation points of the tori orbits under consideration. In §4 we apply these results to deduce Theorem 1.1 and its corollaries. In §5 we use the above mention phenomenon in order to prove that in the case of action of tori of dimension > 2 if the closure of an orbit is not contained in an orbit of a larger torus then it contains curves which approximate arbitrary long pieces of real unipotent orbits. This allows to prove Theorem 1.5 and its corollaries using well-known properties of unipotent flows. The proof of Theorem 1.8 is a result of a careful analyses of the previous arguments in this section. Our number-theoretic

application is proved in §6. §7 contains a specification of Theorem 1.1 and indications for forthcoming works related with the paper.

The main results of the paper have been announced in [T2].

2. Preliminaries

2.1. **Notation.** As usual, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the rational, real and complex numbers, respectively. Also, $\mathbb{R}_+$ (respectively, $\mathbb{R}_-$) is the set of nonnegatives (respectively, nonpositives) real numbers. Let $\mathbb{R}_{>0} = \mathbb{R}_+ \setminus \{0\}$. We denote by $| \ . \ |$ the standard norms on $\mathbb{R}$ and $\mathbb{C}$.

In this paper K is a number field and $K_1, \cdots, K_r$ are the completions of K with respect to the archimedean places of K. We denote by $| \cdot |_i$ *the normalized* valuation on K_i. So, if $x \in K$ and $K_i = \mathbb{R}$ (respectively, $K_i = \mathbb{C}$) then $|x|_i = |\sigma_i(x)|$ (respectively, $|x|_i = |\sigma_i(x)|^2$) where σ_i is the corresponding embedding of K into K_i. Note that $|\mathrm{N}_{K/\mathbb{Q}}(x)| = |x|_1 \cdots |x|_r$, where $\mathrm{N}_{K/\mathbb{Q}}(x)$ is the algebraic norm of x. The elements of K are identified with their images in K_i via the embeddings σ_i. So, if $x \in K$, with some abuse of notation, we write x instead of $\sigma_i(x)$. The exact meaning of x will be always clear from the context.

If R is a ring then R^* is its group of invertible elements.

Let $A = \prod_{i=1}^{r} K_i$ and $A^* = \prod_{i=1}^{r} K_i^*$. A (respectively, A^*) is a topological ring (respectively, topological group) endowed with the product topology. The field K (respectively, the group K^*) is diagonally embedded in A (respectively, A^*). The ring of integers $\mathcal{O}$ of K is a co-compact lattice of A and the group of units $\mathcal{O}^*$ is a discrete subgroup of A^*.

If M is a subset of a topological space X then $\overline{M}$ is the topological closure of M in X. Also, if H is a closed subgroup of a topological group L we denote by H° the connected component of the identity of H. By $\mathcal{N}_L(H)$ (respectively, $\mathcal{Z}_L(H)$) we denote the normalizer (respectively, the centralizer) of H in L.

The notation G_i, G, $G_\mathbb{R}$, D_I, $D_{I,\mathbb{R}}$ have been introduced in the Introduction. The group G is considered as a *real* Lie group.

The diagonal embedding of $\mathrm{SL}(2, K)$ in G will be denoted by G_K. B_K^+, B_K^- and D_K are the groups of upper triangular, lower triangular and diagonal matrices in G_K, respectively. For every $1 \leq i \leq r$ we denote by $G_{i,K}$, $B_{i,K}^+, B_{i,K}^-$ and $D_{i,K}$ the images of G_K, B_K^+, B_K^- and D_K, respectively, under the natural projection $G \to G_i$.

In the course of our considerations one and the same matrix with coefficients from K might be considered, according to the context, as an element from G_K or from $G_{i,K}$. For instance, if $g = (g_1, \cdots, g_r) \in G$ and $g_i \in G_{i,K}$ writing $\pi(g_i)$, where π is the map $G \to G/\Gamma$, $g \mapsto g\Gamma$, we

mean that g_i is considered as an element from G and, therefore, from G_K.

Given a non-empty subset I of $\{1,\cdots,r\}$ we put $A_I^* \stackrel{def}{=} \prod_{i\in I} K_i^*$. Let $d_i : K_i^* \to G_i, x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. We put $d_I \stackrel{def}{=} \prod_{i\in I} d_i$ and $d \stackrel{def}{=} d_{\{1,\cdots,r\}}$. So, $D_I = d_I((A_I^*)^\circ)$.

Let $\mathfrak{g}_i = \mathfrak{sl}(2,K_i)$, $\mathfrak{g} = \prod_{i=1}^{r} \mathfrak{g}_i$, $\mathfrak{g}_K = \mathfrak{sl}(2,K)$ and $\mathfrak{g}_{\mathcal{O}} = \mathfrak{sl}(2,\mathcal{O})$. Fixing a basis of K-rational vectors in $\mathfrak{g}_K$ we denote by $\|\cdot\|_i$ the norm max on $\mathfrak{g}_i$. Since $\mathfrak{g} = \prod_{i=1}^{r} \mathfrak{g}_i$ we can define a norm $\|\cdot\|$ on $\mathfrak{g}$ by $\| \mathbf{x} \| = \max_i \| \mathbf{x}_i \|_i$, $\mathbf{x} = (\mathbf{x}_1,\cdots,\mathbf{x}_r) \in \mathfrak{g}$.

As usual, we denote by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G.

2.2. **Locally divergent orbits.** The following theorem is a very particular case of [T1, Theorem 1.4] (see also [T1, Corollary 1.7]). The paper [T1] is related with [T-W]. Prior to [T-W] Margulis described the divergent orbits for the action of the full diagonal group on the space of lattices of $\mathbb{R}^n$, $n \geq 2$ [T-W, Appendix].

Theorem 2.1. *Let $r \geq 2$, $g = (g_1,\cdots,g_r) \in G$, and I be a non-empty subset of $\{1,\cdots,r\}$. The following assertions hold:*

(a) *If the orbit $D_I\pi(g)$ is closed then either I is a singleton or $I = \{1,\cdots,r\}$;*
(b) *$D_i\pi(g), 1 \leq i \leq r$, is closed (equivalently, divergent) if and only if $g \in \mathcal{N}_G(D_i)G_K$ (equivalently, $g_i \in D_iG_{i,K}$);*
(c) *The following conditions are equivalent:*
   (i) *$D\pi(g)$ is closed and non-compact;*
   (ii) *$D\pi(g)$ is closed and locally divergent;*
   (iii) *$g \in \mathcal{N}_G(D)G_K$.*

We will need the following proposition:

Proposition 2.2. *If $g \in \mathcal{N}_G(D_I)G_K$ then $\overline{D_I\pi(g)} = T\pi(g)$ where T is a torus containing D_I.*

Proof. In view of our assumption $g = g'h$ where $h \in \mathcal{N}_G(D)G_K$ and $g' \in \prod_{i\notin I} G_i$. Let Δ be the stabilizer of $\pi(g)$ in $g'Dg'^{-1}$. It follows from Theorem 2.1(c) that $g'D\pi(h)$ is closed. Since $\overline{D_I\pi(g)} \subset g'Dg'^{-1}\pi(g)$ we get that $\overline{D_I\pi(g)} = T\pi(g)$ where T is the connected component of the identity of $\overline{D_I\Delta}$. $\square$

2.3. **Propositions about the units.** Denote $A^1 = \{(x_1, \cdots, x_r) \in A^* : |x_1|_1 \cdots |x_r|_r = 1\}$. Given a positive integer m we put $\mathcal{O}_m^* = \{\xi^m | \xi \in \mathcal{O}^*\}$.

The following lemma follows easily from the classical fact that $\mathcal{O}^*$ is a lattice in A^1.

Lemma 2.3. (*cf.*[T1, Lemma 3.2]) *Let m be a positive integer. There exists a real $\kappa_m > 1$ with the following property. Let $x = (x_i) \in A^*$ and for each $1 \leq i \leq r$ let a_i be a positive real number such that $\prod_i a_i = \prod_i |x_i|_i$. Then there exists $\xi \in \mathcal{O}_m^*$ such that*

$$\frac{a_i}{\kappa_m} \leq |\xi x_i|_i \leq \kappa_m a_i$$

for all i.

Proposition 2.4. *Let $r \geq 3$, $3 \leq l \leq r$, $I = \{l, \cdots, r\}$ and $p_I : A^* \to A_I^*$ be the natural projection. Denote by H the closure of $p_I(\mathcal{O}^*)$ in A_I^*. Then*

(a) *the projection of H° into each $K_i^*, i \geq l$, is non-trivial;*
(b) *for any real $C > 1$ there exists $\xi \in \mathcal{O}^*$ such that $|\xi|_l > C$ and $|1 - |\xi|_i| < \frac{1}{C}$ for all $i > l$.*

Proof. (a) By Dirichlet's theorem for the units there exists a positive integer m such that $\mathcal{O}_m^*$ is a free abelian group of rank $r-1$. It is clear that H° coincides with the connected component of the closure of $p_I(\mathcal{O}_m^*)$. Since H° is open in H and $\mathcal{O}_m^*$ is diagonally embedded in H it is enough to show that $H^\circ \neq \{1\}$. Suppose that $H^\circ = \{1\}$. Then H is a discrete subgroup of A_I^* containing a free subgroup of rank $r-1$. This is a contradiction because $l \geq 3$ and A_I^* is a direct product of a compact group and $\mathbb{Z}^{r-l+1}$.

(b) Consider the logarithmic representation of the group of units $\log_S : \mathcal{O}^* \to \mathbb{R}^r, \theta \mapsto (\log|\theta|_1, \cdots, \log|\theta|_r)$ (see [We]). According to the Dirichlet theorem $\log_S(\mathcal{O}^*)$ is a lattice in the hyperplane $L = \{(x_1, \cdots, x_r) \in \mathbb{R}^r : x_1 + x_2 + \cdots + x_r = 0\}$. Let $\psi : L \to \mathbb{R}^{r-1}, (x_1, \cdots, x_r) \mapsto (x_2, \cdots, x_r)$. Then $\psi(\log_S(\mathcal{O}^*))$ is a lattice in $\mathbb{R}^{r-1}$ with co-volume equal to a positive real V. For every natural n, we denote

$$B_n = \{(x_2, \cdots, x_r) \in \mathbb{R}^{r-1} : |x_i| \leq \frac{1}{n} \text{ if } i \neq l \text{ and } |x_l| \leq n^{r-2}V\}.$$

By Minkowski's lemma there exists a $\xi_n \in \mathcal{O}^*$ such that $\psi(\log_S(\xi_n)) \in B_n \setminus \{0\}$. If the sequence $|\xi_n|_l$ is unbounded from above then we can choose $\xi = \xi_n$ with n large enough. Let $|\xi_n|_l < C$ where C is a constant. Since $\psi(\log_S(\mathcal{O}^*))$ is discrete this implies the existence of a unit η of

infinite order such that $|\eta|_l > 1$ and $|\eta|_i = 1$ if $i \neq l$ and $i > 1$. Hence we can choose $\xi = \eta^m$ with m sufficiently large. $\square$

Proposition 2.5. *Let $p_l : A^* \to K_l^*$, $1 \leq l \leq r$, be the natural projection. Assume that $K_l = \mathbb{C}$ and that the connected component of the identity of $\overline{p_l(\mathcal{O}^*)}$ coincides with $\mathbb{R}_{>0}$. Then K is a* CM*-field.*

Proof. There exists a positive integer m such that $\overline{p_l(\mathcal{O}_m^*)} = \mathbb{R}_{>0}$. Denote by F the subfield of K generated over $\mathbb{Q}$ by all $\theta \in \mathcal{O}_m^*$ and denote by $\mathcal{O}_F^*$ the group of units of F. Let s, respectively t, be the number of real, respectively complex, places of K and let s_1, respectively t_1, be the number of real, respectively complex, places of F. By Dirichlet's theorem $\mathcal{O}_m^*$ is a free group of rank $s + t - 1$. Since $\mathcal{O}_m^* \subset \mathcal{O}_F^* \subset \mathcal{O}^*$ and the group of principal units of F is free of rank $s_1 + t_1 - 1$ we have

$$r - 1 = s + t - 1 = s_1 + t_1 - 1.$$

Let n be the degree of K over F. Since $s + 2t$ is the degree of K over $\mathbb{Q}$ and $s_1 + 2t_1$ is the degree of F over $\mathbb{Q}$ we get

$$s + 2t = n(s_1 + 2t_1) \Leftrightarrow r + t = n(r + t_1) \Leftrightarrow$$
$$(n-1)r = t - t_1 n \Leftrightarrow (n-1)(t+s) = t - t_1 n.$$

Since $n > 1$ the last equality implies that $s = t_1 = 0$ and $n = 2$ proving the proposition. $\square$

Example.[2] There are non-CM fields such that the connected components of the identity of $\overline{p_l(\mathcal{O}^*)}$ are 1-dimensional subgroups of $\mathbb{C}^*$ different from $\mathbb{R}_{>0}$. Such fields need special treatment in the course of the proof of Proposition 5.1(a) below. An example of this type is provided by the field $K = \mathbb{Q}(\alpha)$ where α is a root of the equation $(x + \frac{1}{x})^2 - 2(x + \frac{1}{x}) - 1 = 0$. The field K has two real and one (up to conjugation) complex completions. If $K_3 = \mathbb{C}$ then it is easy to see that $\overline{p_3(\mathcal{O}^*)}^\circ$ coincides with the unit circle.

3. Accumulations points for locally divergent orbits

Up to the end of the paper $D_I\pi(g)$ will denote a locally divergent orbit. In view of Theorem 2.1(b), *we may (and will) assume without loss of generality that $g = (g_1, \cdots, g_r)$ with $g_i \in G_{i,K}$ whenever $i \in I$.*

The following lemma is an easy consequence from the commensurability of Γ and $h\Gamma h^{-1}$ when $h \in G_K$.

Lemma 3.1. *Let $h \in G_K$. The following assertions hold:*

[2] This example is essentially due to Yves Benoist.

(a) *There exists a positive integer* m *such that* $d(\xi)\pi(h) = \pi(h)$ *for all* $\xi \in \mathcal{O}_m^*$;
(b) *If* $\{\pi(g_i)\}$ *is a converging sequence in* G/Γ *then there exists a converging subsequence of* $\{\pi(g_ih)\}$;
(c) *If* $\overline{D_I\pi(g)} = G/\Gamma$ *then* $\overline{D_I\pi(gh)} = G/\Gamma$.

Proposition 3.2. *Let* $I = \{1,2\}$ *and* $(s_k, t_k) \in K_1^* \times K_2^*$ *be a sequence such that* $|\log|s_k|_1| + |\log|t_k|_2| \underset{k}{\to} \infty$ *and* $d_I(s_k,t_k)\pi(g)$ *converges to an element from* G/Γ. *Then:*

(a) *There exists a constant* $C > 1$ *such that* $-C < |\log|s_k|_1| - |\log|t_k|_2| < C$;
(b) *Let* $|s_k|_1 \to \infty$, $|t_k|_2 \to 0$. *Then* $g_1g_2^{-1} = b_-b_+^{-1}$, *where* $b_- \in B_K^-$ *and* $b_+ \in B_K^+$.

Proof.(a) The remaining cases being analogous, it is enough to consider the case when $|s_k|_1 \to \infty$ and $\sup\limits_k \frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} < \infty$.

Assume on the contrary that (a) is false. Then $\frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} \underset{k}{\to} 0$. It is well known that for every $h \in G_K$ $\mathrm{Ad}(h)\mathfrak{g}_{\mathcal{O}}$ is commensurable with $\mathfrak{g}_{\mathcal{O}}$. Since $g_1 \in G_K$ this implies the existence of $\mathbf{u} \in \mathrm{Ad}(g)\mathfrak{g}_{\mathcal{O}}, \mathbf{u} \neq 0$, such that $\mathrm{pr}_1(\mathbf{u})$ is a lower triangular nilpotent matrix where pr_1 is the projection of $\mathfrak{g}$ to $\mathfrak{g}_1$. Recall that $\mathfrak{g} = \prod\limits_{i=1}^r \mathfrak{g}_i$. Let $\mathrm{Ad}(d_I(s_k,t_k))(\mathbf{u}) = (\mathbf{u}_1^{(k)}, \cdots, \mathbf{u}_r^{(k)}) \in \mathfrak{g}$. Since $\frac{\max\{|t_k|_2, |t_k|_2^{-1}\}}{|s_k|_1} \underset{k}{\to} 0$ and $\mathrm{Ad}(d_I(s_k,t_k))$ is acting by conjugation on the elements from $\mathfrak{g}$, we see that $\|\mathbf{u}_1^{(k)}\|_1 \cdots \|\mathbf{u}_r^{(k)}\|_r \underset{k}{\to} 0$. In view of Lemma 2.3, there exists a sequence $\xi_k \in \mathcal{O}^*$ such that $\| \mathrm{Ad}(d_I(s_k,t_k))(\xi_k\mathbf{u}) \| = \|(\xi_k\mathbf{u}_1^{(k)}, \cdots, \xi_k\mathbf{u}_r^{(k)})\| \underset{k}{\to} 0$. By Mahler's compactness criterion $d_I(s_k,t_k)\pi(g)$ tends to infinity which is a contradiction.

(b) By Bruhat decomposition

$$G_K = B_K^+ \cup B_K^+\omega B_K^+ = \omega B_K^+ \cup B_K^-B_K^+,$$

where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Suppose on the contrary that $g_1g_2^{-1} \in \omega B_K^+$. Shifting g from the right by g_2^{-1} and from the left by a suitable element from $\mathcal{Z}_G(D_I)$ we may (and will) assume with no loss of generality (see Lemma 3.1(b)) that $g_1 = \omega u^+(\alpha)$, where $u^+(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in K$, and $g_i = e$ for all $i > 1$. In view of (a), there exists a constant $C > 1$ such that

$\frac{1}{C} < |s_k|_1 \cdot |t_k|_2 < C$. Now using Lemma 3.1(a) and Lemma 2.3 we find a sequence $\xi_k \in \mathcal{O}^*$ and a positif constant κ such that $d(\xi_k)\pi(u^+(\alpha)) = \pi(u^+(\alpha))$, $\frac{1}{\kappa} < \frac{|s_k|_1}{|\xi_k|_1} < \kappa$, $\frac{1}{\kappa} < \frac{|t_k|_2}{|\xi_k|_2} < \kappa$ and $\frac{1}{\kappa} < |\xi_k|_i < \kappa$ for all $i > 2$. Then $(s_k, t_k, 1, \cdots, 1) = \xi_k a_k$ where $a_k \in A^*$ is a bounded sequence. Passing to a subsequence we can suppose that a_k converges to an element from A^*. Note that $d(\xi_k)\pi(g)$ converges in G/Γ because $d_I(s_k, t_k)\pi(g)$ does.

By an easy computation:

$$\begin{aligned}
&d(\xi_k)\pi(g) = d(\xi_k)(\omega u^+(\alpha), e, \cdots, e)\pi(e) = \\
&d(\xi_k)(\omega, u^+(-\alpha), \cdots, u^+(-\alpha))\pi(u^+(\alpha)) = \\
&d(\xi_k)(\omega, u^+(-\alpha), \cdots, u^+(-\alpha))d(\xi_k^{-1})\pi(u^+(\alpha)) = \\
&(\omega, u^+(-\alpha\xi_k^2), \cdots, u^+(-\alpha\xi_k^2))(d_1(\xi_k^{-2}), e, \cdots, e)\pi(u^+(\alpha)).
\end{aligned}$$

In view of the choice of ξ_k we have that $|\xi_k|_1 \to \infty$ and $|\xi_k|_2 \to 0$. Hence $|\xi_k|_i < \kappa$ if $i \geq 2$ and k is sufficiently large. So, after passing to a subsequence, $(\omega, u^+(-\alpha\xi_k^2), \cdots, u^+(-\alpha\xi_k^2))$ converges to an element from G and $d_1(\xi_k^{-2})$ tends to infinity. The latter contradicts the convergence of the sequence $d(\xi_k)\pi(g)$. Therefore, $g_1 g_2^{-1} \in B_K^- B_K^+$. $\square$

Proposition 3.3. *Let $I = \{1, \cdots, l\}$ where $1 < l \leq r$, $g_1 = \cdots = g_{l-1}$ and $g_1 g_l^{-1} = b_- b_+^{-1}$ where $b_- \in B_K^-$ and $b_+ \in B_K^+$. Denote $h = b_-^{-1} g_1 = b_+^{-1} g_l$. Then we have the following:*

(a) $(h, \cdots, h, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I \pi(g)}$;

(b) *Let $s_k = (s_k^{(1)}, \cdots, s_k^{(l)}) \in A_I^*$ be such that $|s_k^{(i)}|_i \underset{k}{\to} \infty$ for all $1 \leq i < l$, $|s_k^{(l)}|_l \underset{k}{\to} 0$ and $\frac{1}{C} < |s_k^{(1)}|_1 \cdots |s_k^{(l)}|_l < C$, where C is a positive constant. Then $d_I(s_k)\pi(g)$ admits a converging subsequence and the limit of every such subsequence belongs to $\overline{D_I \pi((h, \cdots, h, g_{l+1}, \cdots, g_r))}$.*

Proof. Fix m such that $d(\xi)\pi(h) = \pi(h)$ for all $\xi \in \mathcal{O}_m^*$. With s_k as in the formulation of (b), in view of Lemma 2.3 there exists a sequence $\xi_k \in \mathcal{O}_m^*$ and a constant $C_1 > 1$ such that $\frac{1}{C_1} < |s_k^{(i)} \xi_k^{-1}|_i < C_1$ if $1 \leq i \leq l$ and $\frac{1}{C_1} < |\xi_k|_i < C_1$ if $i > l$. Put $a_k = (\underbrace{\xi_k, \cdots, \xi_k}_{l}, \underbrace{1, \cdots, 1}_{r-l})$ and $a'_k = (\underbrace{1, \cdots, 1}_{l}, \underbrace{\xi_k, \cdots, \xi_k}_{r-l})$. Passing to a subsequence we may assume that $a'_k \to a'$ where $a' \in A^*$. In view of the choice of ξ_k and the

proposition hypothesis, we get

$$\lim_k d_i(\xi_k)b_- d_i(\xi_k)^{-1} = t_-, \ \forall\ 1 \le i < l,$$

and

$$\lim_k d_l(\xi_k)b_+ d_l(\xi_k)^{-1} = t_+,$$

where t_- and $t_+ \in D_K$. It is enough to prove (b) in the particular case when $s_k^{(i)} = t_-^{-1}\xi_k$, $1 \le i < l$, and $s_k^{(l)} = t_+^{-1}\xi_k$.

Since $d(\xi_k)\pi(h) = \pi(h)$, we get

$$d_I(s_k)\pi(g) = (d_1(t_-^{-1}\xi_k)b_-, \cdots, d_l(t_+^{-1}\xi_k)b_+, g_{l+1}h^{-1}, \cdots, g_r h^{-1})\pi(h) =$$
$$(d_1(t_-^{-1}\xi_k)b_-, \cdots, d_l(t_+^{-1}\xi_k)b_+, g_{l+1}h^{-1}, \cdots, g_r h^{-1}))d({a_k}^{-1})d({a'_k}^{-1})\pi(h).$$

Therefore

(1)
$$\lim_k d_I(s_k)\pi(g) = (e, \cdots, g_{l+1}h^{-1}, \cdots, g_r h^{-1})d(a'^{-1})\pi(h) \in \overline{D_I\pi(g)}.$$

Since

$$d(a_k)^{-1}\pi(h) = d(a_k)^{-1}d(\xi_k)\pi(h) = d(a'_k)\pi(h) \to d(a')\pi(h),$$

multiplying (1) by $d(a_k)^{-1}$ and passing to a limit, we obtain that

$$(h, \cdots, h, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I\pi(g)}.$$

Since a sequence s_k with properties as in the formulation of (b) always exists, the above proves (a). In order to complete the proof of (b) it remains to note that

$$\lim_k d_I(s_k)\pi(g) = \lim_k d(a_k)\pi((h, \cdots, h, g_{l+1}, \cdots, g_r)).$$

$\square$

Let $h \in G_K$. A pair $(\sigma_1, \sigma_2) \in \{0,1\}^2$ is called *admissible with respect to h* if $\omega^{\sigma_1}h\omega^{\sigma_2} \in B_K^- B_K^+$, where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The following lemma can be proved by a simple calculation.

Lemma 3.4. *With the above notation, (σ_1, σ_2) is admissible with respect to $h = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ if and only if $m_{1+\sigma_1, 1+\sigma_2} \neq 0$.*

It is clear that $h \in \mathcal{N}_{G_K}(D_K)$ if and only if the number of admissible pairs is equal to 2.

Proposition 3.5. *Let* $I = \{1, \cdots, l\}$*, where* $1 < l < r$*,* $g_1 = \cdots = g_{l-1}$ *and* $g_1 g_l^{-1} \notin \mathcal{N}_{G_K}(D_K)$*. Then* $\overline{D_I \pi(g)}$ *contains a point*

$$(\underbrace{nh, \cdots, nh}_{l-1}, h, g_{l+1}, \cdots, g_r)\pi(e),$$

where $n \in \mathcal{N}_{G_K}(D_K), h \in G_K$ *and* $hg_{l+1}^{-1} \notin \mathcal{N}_{G_K}(D_K)$*.*

Proof. If the pair (σ_1, σ_2) is admissible with respect to $g_1 g_l^{-1}$ then $\omega^{\sigma_1} g_1 (\omega^{\sigma_2} g_l)^{-1} = b_- b_+^{-1}$, where $b_- \in B_K^-$ and $b_+ \in B_K^+$, and we put $h_{\sigma_1,\sigma_2} = b_-^{-1}\omega^{\sigma_1} g_1 = b_+^{-1}\omega^{\sigma_2} g_l$. Shifting $\pi(g)$ from the left by

$$(\underbrace{\omega^{\sigma_1}, \cdots, \omega^{\sigma_1}}_{l-1}, \omega^{\sigma_2}, e, \cdots, e)$$

and applying Proposition 3.3(a) we get that

$$(\underbrace{\omega^{\sigma_1} h_{\sigma_1,\sigma_2}, \cdots, \omega^{\sigma_1} h_{\sigma_1,\sigma_2}}_{l-1}, \omega^{\sigma_2} h_{\sigma_1,\sigma_2}, g_{l+1}, \cdots, g_r)\pi(e) \in \overline{D_I \pi(g)}.$$

It remains to prove that (σ_1, σ_2) can be chosen in such a way that $h_{\sigma_1,\sigma_2} g_{l+1}^{-1} \notin \mathcal{N}_{G_K}(D_K)$. Since $g_1 g_l^{-1} \notin \mathcal{N}_{G_K}(D_K)$, in view of Lemma 3.4 there are at least 3 admissible pairs with respect to $g_1 g_l^{-1}$. Shifting g from the left by an appropriate element from $\mathcal{N}_{G_K}(D_K)$, we may assume that $(0, 0)$ and $(0, 1)$ are admissible pairs. Then

$$h_{0,0} = b'^{-1}_- g_1 = b'^{-1}_+ g_2 \text{ and } h_{1,0} = \widetilde{b}_-^{-1}\omega g_1 = \widetilde{b}_+^{-1} g_2,$$

where $b'_-, \widetilde{b}_- \in B_K^-$ and $b'_+, \widetilde{b}_+ \in B_K^+$. Suppose on the contrary that both $h_{0,0} g_{l+1}^{-1}$ and $h_{1,0} g_{l+1}^{-1} \in \mathcal{N}_{G_K}(D_K)$. In view of the above expressions for $h_{0,0}$ and $h_{1,0}$, we obtain

$$h_{0,0} h_{1,0}^{-1} \in \mathcal{N}_{G_K}(D_K) \cap B_K^+ \cap B_K^- \omega B_K^-.$$

This is a contradiction because $\mathcal{N}_{G_K}(D_K) \cap B_K^+ = D_K$ and $D_K \cap B_K^- \omega B_K^- = \emptyset$. □

4. Proofs of Theorem 1.1 and Corollaries 1.2 and 1.4

4.1. **Proof of Theorem 1.1.** We suppose that $I = \{1, 2\}$. It follows from Proposition 2.2 that $g_1 g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$. Let $(s_k, t_k) \in K_1^* \times K_2^*$ be an unbounded sequence such that $d_I(s_k, t_k)\pi(g)$ converges. In view of Proposition 3.2(a) there exists a positive constant C such that $-C < |\log|s_k|_1| - |\log|t_k|_2| < C$. Passing to a subsequence there exist σ_1 and $\sigma_2 \in \{0, 1\}$ such that $\omega^{\sigma_1} d_1(s_k)\omega^{-\sigma_1} = d_1(s'_k)$, $\omega^{\sigma_2} d_2(t_k)\omega^{-\sigma_2} = d_2(t'_k)$ where $|s'_k|_1 \to \infty$ and $|t'_k|_2 \to 0$. Let $g' = (\omega^{\sigma_1} g_1, \omega^{\sigma_2} g_2, g_3, \cdots, g_r)$.

It follows from Proposition 3.2(b) that $\omega^{\sigma_1}g_1(\omega^{\sigma_2}g_2)^{-1} = b_-b_+^{-1} \in B_K^-B_K^+$, i.e., (σ_1, σ_2) is an admissible pair with respect to $g_1g_2^{-1}$. Let

$$h_{\sigma_1,\sigma_2} = b_-^{-1}\omega^{\sigma_1}g_1 = b_+^{-1}\omega^{\sigma_2}g_2. \tag{2}$$

Using Proposition 3.3(b) we get:

$$\lim_k d_I(s'_k, t'_k)\pi(g') \in \overline{D_I\pi((h_{\sigma_1,\sigma_2}, h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))}.$$

Therefore

$$\lim_k d_I(s_k, t_k)\pi(g) \in \overline{D_I\pi((\omega^{\sigma_1}h_{\sigma_1,\sigma_2}, \omega^{\sigma_2}h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))}.$$

It follows that

$$\overline{D_I\pi(g)} \subset D_I\pi(g) \cup \cup_{(\sigma_1,\sigma_2)\in M}\overline{D_I\pi((\omega^{\sigma_1}h_{\sigma_1,\sigma_2}, \omega^{\sigma_2}h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))},$$

where M is the set of all admissible pairs with respect to $g_1g_2^{-1}$. On the other hand, using Proposition 3.3(a) we get:

$$\overline{D_I\pi(g)} = D_I\pi(g) \cup \cup_{(\sigma_1,\sigma_2)\in M}\overline{D_I\pi((\omega^{\sigma_1}h_{\sigma_1,\sigma_2}, \omega^{\sigma_2}h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))}. \tag{3}$$

Note that

$$\overline{D_I\pi((\omega^{\sigma_1}h_{\sigma_1,\sigma_2}, \omega^{\sigma_2}h_{\sigma_1,\sigma_2}, g_3, \cdots, g_r))} = (\omega^{\sigma_1}, \omega^{\sigma_2}, g_3h_{\sigma_1,\sigma_2}^{-1}, \cdots, g_rh_{\sigma_1,\sigma_2}^{-1})\overline{D_I\pi(h_{\sigma_1,\sigma_2})}.$$

Since $D\pi(h_{\sigma_1,\sigma_2})$ is a closed locally divergent orbit, each of the closures in the right hand side of (3) is a non-compact orbit of a torus containing D_I. It remain to see that at least two of these orbits are different.

Since $g_1g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$ there exists $\sigma \in \{0, 1\}$ such that $(\sigma, 0)$ and $(\sigma, 1) \in M$. Suppose on the contrary that

$$\overline{D_I\pi(\omega^\sigma h_{\sigma,0}, h_{\sigma,0}, g_3, \cdots, g_r)} = \overline{D_I\pi(\omega^\sigma h_{\sigma,1}, \omega h_{\sigma,1}, g_3, \cdots, g_r)}.$$

There exist tori T and T' containing D_I such that

$$T\pi((h_{\sigma,0}, h_{\sigma,0}, g_3, \cdots, g_r)) = T'\pi((h_{\sigma,1}, \omega h_{\sigma,1}, g_3, \cdots, g_r)).$$

Then

$$h_{\sigma,0} = th_{\sigma,1}\gamma = t'\omega h_{\sigma,1}\gamma,$$

where $t, t' \in D_K$ and $\gamma \in \Gamma$. Hence

$$\omega = t't^{-1}$$

which is a contradiction. $\square$

4.2. **Proof of Corollary 1.2.** We use the notation from the formulations of Theorem 1.1 and the Corollary. Let us show that both $D_I\pi(g)$ and $D_{I,\mathbb{R}}\pi(g)$ are open and proper in their closures. Note that if $D_I\pi(g) \cap T_i\pi(h_i) \neq \emptyset$ for some $1 \leq i \leq s$ then $\overline{D_I\pi(g)} \subset T_i\pi(h_i)$ which contradicts the fact that $s \geq 2$. Therefore, the orbit $D_I\pi(g)$ is open and proper in its closure. Suppose that there exists i such that $\overline{D_{I,\mathbb{R}}\pi(g)} \cap T_i\pi(h_i) = \emptyset$. Since $T_i \supset D_I$ this implies that $\overline{D_I\pi(g)} \cap T_i\pi(h_i) = \emptyset$ which is a contradiction. Therefore, $\overline{D_{I,\mathbb{R}}\pi(g)} \cap T_i\pi(h_i) \neq \emptyset$ for every $1 \leq i \leq s$. So, the orbit $D_{I,\mathbb{R}}\pi(g)$ is open and proper in its closure too. Now if, supposing the contrary, $\overline{T\pi(g)} = H\pi(g)$ for some closed subgroup H then H is locally homeomorphic to T. Since T is generated by any neighborhood of the identity, $T\pi(g)$ must be closed. This is a contradiction completing our proof. $\square$

4.3. **Proof of Corollary 1.4.** (a) It is enough to show that f represents over $\mathbb{Q}$ a quadratic form f_1 of 4 variables such that $\mathrm{rank}_{\mathbb{Q}} f_1 = 1$ and $\mathrm{rank}_{\mathbb{R}} f_1 = 2$. Indeed, in this case we may suppose without loss of generality that $f = f_1 + f_2$ where f_2 is a quadratic form over the rationals of $n-4$ variables. Remark that $\mathrm{SO}(f_1,\mathbb{R}) \times \mathrm{SO}(f_2,\mathbb{R})$ is a $\mathbb{Q}$-subgroup of $\mathrm{SO}(f,\mathbb{R})$ and $(\mathrm{SO}(f_1,\mathbb{R}) \times \mathrm{SO}(f_2,\mathbb{R})) \cap \mathrm{SO}(f,\mathbb{Z})$ is commensurable with $\mathrm{SO}(f_1,\mathbb{Z}) \times \mathrm{SO}(f_2,\mathbb{Z})$. It is known that $\mathrm{SO}(f_1,\mathbb{R}) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and that $\mathrm{SO}(f_1,\mathbb{Z})$ corresponds under this isomorphisme to the diagonal embedding of $\mathrm{PSL}(2,\mathbb{Z}[\sqrt{d}])$ into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, where d is the discriminant of f_1 [A, Theorems 5.21 and 5.22]. If T_1 is a maximal $\mathbb{R}$-split torus of $\mathrm{SO}(f_1,\mathbb{R})$ and T_2 is a maximal $\mathbb{R}$-split torus of $\mathrm{SO}(f_2,\mathbb{R})$ then $T = T_1 \times T_2$ is a maximal $\mathbb{R}$-split torus of $\mathrm{SO}(f,\mathbb{R})$. Now, if we choose $g_1 \in \mathrm{SO}(f_1,\mathbb{R})$ in such a way that the boundary of the closure of the orbit $T_1 g_1 \mathrm{SO}(f_1,\mathbb{Z})$ consists of 4 different T_1-orbits (Proposition 7.1) and if we choose $g_2 \in \mathrm{SO}(f_2,\mathbb{R})$ in such a way that the orbit $T_2 g_2 \mathrm{SO}(f_2,\mathbb{Z})$ is closed (see, for example, [T1, Proposition 4.2]) then $\overline{T\pi_\circ(g)}$, where $g = (g_1, g_2)$, is as required.

Let us prove that f represents over $\mathbb{Q}$ a quadratic form f_1 with the above mentioned properties. Since $\mathrm{rank}_{\mathbb{Q}} f \geq 1$ and $\mathrm{rank}_{\mathbb{R}} f \geq 2$ the form f is $\mathbb{Q}$-equivalent to a form $x_1x_2 + x_3^2 - ax_4^2 - bx_5^2 + f'(x_6, \cdots, x_n)$ where a and b are rational numbers such that $a \cdot b \neq 0$ and $b > 0$ (see [C]). If $b \notin \mathbb{Q}^2$ then we can choose $f_1 = x_1x_2 + x_3^2 - bx_5^2$. Suppose that $b \in \mathbb{Q}^2$. Then the form $x_3^2 - bx_5^2$ represents a rational number α such that $a \cdot \alpha \notin \mathbb{Q}^2$ and $a \cdot \alpha > 0$. Therefore f represents a form f_1 which is $\mathbb{Q}$-equivalent to $x_1x_2 + \alpha x_3^2 - ax_4^2$.

(b)Let G and Γ be as in the formulation of Corollary 1.3 with K a real quadratic number field. Using Weil's restriction of scalars [Z, Ch.6], we get an injective homomorphism $\mathrm{R}_{K/\mathbb{Q}} : G \to \mathrm{SL}(4, \mathbb{R})$ such that $\mathrm{R}_{K/\mathbb{Q}}(\Gamma) = \mathrm{R}_{K/\mathbb{Q}}(G) \cap \mathrm{SL}(4, \mathbb{Z})$. Let $\phi : G \to \mathrm{SL}(n, \mathbb{R}), g \mapsto \begin{pmatrix} \mathrm{R}_{K/\mathbb{Q}}(g) & 0 \\ 0 & \mathrm{I}_{n-4} \end{pmatrix}$, where I_{n-4} is the identity matrix of rank $n-4$. Further on we identify G, D and Γ with $\phi(G)$, $\phi(D)$ and $\phi(\Gamma)$, respectively. Let F be the connected component of the identity of the centralizer of G in $\mathrm{SL}(n, \mathbb{R})$. It is clear that F is a real reductive $\mathbb{Q}$-group, $G \cap F$ is finite and $L = GF$ is a reductive group of real rank $n-1$. Put $\Gamma_F = F \cap \mathrm{SL}(n, \mathbb{Z})$. Since L is a reductive $\mathbb{Q}$-group the orbit $L\Gamma$ is closed in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ ([T1, Proposition 4.2]). Therefore the map $G/\Gamma \times F/\Gamma_F \to \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$, $(g\Gamma, h\Gamma_F) \mapsto \pi_\circ(gh)$, is proper with finite fibers. Let T_F be a maximal $\mathbb{R}$-split torus in F and $h \in F$ be such that $T_F h \Gamma_F$ is dense in F. Choose $g \in G$ such that the boundary of $D\pi(g)$ consists of four pairwise different closed D-orbits (Proposition 7.1). Denote $T' = DT_F$. It follows from the above that the boundary of $T'\pi_\circ(gh)$ consists of four pairwise different closed T'-orbits. In order to complete the proof it remains to note that T and T' are conjugated in $\mathrm{SL}(n, \mathbb{R})$. $\square$

5. Closures of D_I-orbits when $\#I > 2$

5.1. **Main Proposition.** If K is a CM-field we denote by F the totally real subfield of K of index 2. In this case we denote by F_i the completion of F with respect to the valuation $|\ .\ |_i$ on K_i and by $\mathcal{O}_F$ the ring of integers of F. We put $A_F = \prod_i F_i$.

In this section $I = \{1, \cdots, l\}$ where $3 \leq l \leq r$.

Proposition 5.1. *Let* $h = (e, \cdots, \underbrace{u_l^-(\beta)u_l^+(\alpha)}_{l}, \cdots, e) \in G$ *where* $u_l^-(\beta) = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$, $u_l^+(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in K^*$ *and* $\beta \in K_l$. *The following assertions hold:*

(a) *If K is not a* CM-*field then* $\overline{D_I\pi(h)} = G/\Gamma$;
(b) *Let K be a* CM-*field and d_α be an element in D such that* $d_\alpha^2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. *Then* $\overline{D_{I,\mathbb{R}}\pi(h)} \supset d_\alpha G_{\mathbb{R}} d_\alpha^{-1}\pi(e)$ *and* $d_\alpha G_{\mathbb{R}} d_\alpha^{-1}\pi(e)$ *is closed.*

In order to prove the proposition we need the following lemma.

Lemma 5.2. *Let K be a* CM*-field and $\alpha \in K^*$. Then*
$$\overline{F_l\alpha + \mathcal{O}} = A_F\alpha + \mathcal{O}.$$

Proof. Let n be a positive integer such that $n\alpha \in \mathcal{O}$. By the classical strong approximation theorem $\overline{F_l + \mathcal{O}_F} = A_F$. Since $A_F \cap \mathcal{O} = \mathcal{O}_F$ we have that $A_F + \mathcal{O}$ is closed in A (cf.[R, 1.13]) and, therefore,
$$\overline{F_l + \mathcal{O}} = A_F + \mathcal{O}.$$
Put
$$L = \overline{F_l\alpha + \mathcal{O}n\alpha} = A_F\alpha + \mathcal{O}n\alpha.$$
Since $\mathcal{O}n\alpha$ has finite index in $\mathcal{O}$, $L \cap \mathcal{O}$ is a lattice in L. Hence $L + \mathcal{O}$ is a closed subgroup of A which, in view of the definition of L, completes the proof. □

Proof of Proposition 5.1. Note that $U^+(A)\pi(e)$ is closed and homeomorphic to $A/\mathcal{O}$. (We denote by $U^+(A)$ the group of A-points of the upper unipotent subgroup of G.) This implies that $u_l^+(K_l)\pi(e)$ is dense in $U^+(A)\pi(e)$ and, when K is a CM-field, it follows from Lemma 5.2 that $u_l^+(F_l\alpha)\pi(e)$ is dense in the closed set $U^+(A_F\alpha)\pi(e)$.

Further the proof proceeds in several steps.

Step 1. As in the formulation of Proposition 2.4, let H be the closure of the projection of $\mathcal{O}^*$ into $K_l^* \times \cdots \times K_r^*$. Denote by $p_j : A^* \to K_j^*$, $l \leq j \leq r$, the natural projections. We will consider the case (a) (when K is not a CM-field) and the case (b) (when K is a CM-field) in a parallel way. Using Propositions 2.4(a) and 2.5, for every positive integer m we fix in H° a compact neighborhood H_m of 1 with the following properties: (i) $1 - \frac{1}{m} < |p_j(x)|_j < 1 + \frac{1}{m}$ for all $j \geq l$ and all $x \in H_m$ and, (ii) $p_l(H_m) = \{e^{(a_m + \imath b_m)t} : t \in [-\frac{1}{m}, \frac{1}{m}]\}$, where $\imath = \sqrt{-1}$ and a_m and b_m are real numbers such that $b_m \neq 0$ (resp. $b_m = 0$ and $a_m \neq 0$) if $K_l = \mathbb{C}$ and we are in case (a) (resp. if otherwise). In view of Proposition 2.4(b) there exists a sequence $y_n \in \mathcal{O}^*$ such that $y_n \in \mathcal{O}_F^*$ in case (b), $|p_l(y_n)|_l > n$ and $1 - \frac{1}{n} < |p_j(y_n)|_j < 1 + \frac{1}{n}$ for all $j > l$.

Step 2. Denote
$$L_{mn} = \{x^2 : x \in y_nH_m\}.$$
Let W_ε be the ε-neighborhood of 0 in A and $W_{\varepsilon,F}$ be the ε-neighborhood of 0 in A_F. We claim that given m for every $\varepsilon > 0$ there exists a constant $n_\circ$ such that if $n > n_\circ$ then
$$A = W_\varepsilon + p_l(L_{mn}) + \mathcal{O} \tag{4}$$
in case (a), and
$$A_F = W_{\varepsilon,F} + p_l(L_{mn}) + \mathcal{O}_F \tag{5}$$

in case (b).

Note that the projections of K_l into $A/\mathcal{O}$ and of F_l into $A_F/\mathcal{O}_F$ are dense and equidistributed. Since $|p_l(y_n)|_l > n$ this implies the claim in case (b) and in case (a) when $K_l = \mathbb{R}$.

Consider the case (a) when $K_l = \mathbb{C}$. If $\theta \in [0, 2\pi)$ we put $\mathbb{R}_\theta = e^{\imath\theta}\mathbb{R}$ and if $a < b$ we put $[a, b]_\theta = e^{\imath\phi}[a, b]$ where $\mathbb{R}$ stands for the subfield of reals in K_l. Since $\overline{K_l + \mathcal{O}} = A$ it is easy to see that for almost all $\theta \in [0, 2\pi)$ we have that $\overline{\mathbb{R}_\theta + \mathcal{O}} = A$ and, moreover, given $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that if $b - a > c_\varepsilon$ then

$$A = W_\varepsilon + z + [a, b]_\theta + \mathcal{O},\ \forall z \in A.$$

Now let $p_l(y_n) = r_n e^{\imath\frac{\psi_n}{2}}$ and $\psi_n \underset{n}{\to} \psi$. Since the real b_m in the definition of H_m is different from 0 there exists $\frac{\theta}{2} \in (-\frac{1}{m}, \frac{1}{m})$ such that $\overline{\mathbb{R}_{\theta+\psi} + \mathcal{O}} = A$. Remark that since $r_n \to +\infty$ the curvatures at the points of the plane curve $p_l(L_{mn}) \subset \mathbb{C}$ are tending uniformly to 0 when $n \to \infty$. Therefore for every real $\beta > 0$ end every $\varepsilon > 0$ there exist a positive integer $n_\circ$ such that for every $n > n_\circ$ there exists a $z \in K_l$ such that the points of the segment $z + [0, \beta]_{\theta+\psi}$ are ε-close to $p_l(L_{nm})$. This implies the claim.

Step 3. Since $d(\xi^{-1})\pi(e) = \pi(e)$ for every $\xi \in \mathcal{O}^*$ we have that $(e, \cdots, u_l^-(\xi^{-2}\beta)u_l^+(\xi^2\alpha), d_{l+1}(\xi)^{-1}, \cdots, d_r(\xi)^{-1})\pi(e)$ belongs to $D_I\pi(h)$ (respectively, $D_{I,\mathbb{R}}\pi(h)$) if K is not (respectively, is) a CM-field. Put

$$X_{mn} \overset{def}{=} \{(e, \cdots, \underbrace{u_l^-(x^{-2}\beta)u_l^+(x^2\alpha)}_{l}, \cdots, d_r(x)^{-1})\pi(e) : x \in y_n H_m\}$$

Since $y_n H_m \cap \mathcal{O}^*$ is dense in $y_n H_m$,

$$X_{mn} \subset \overline{D_I\pi(h)} \tag{6}$$

in case (a), and

$$X_{mn} \subset \overline{D_{I,\mathbb{R}}\pi(h)} \tag{7}$$

in case (b). Using the commensurability of $\mathcal{O}$ and $\mathcal{O}\alpha$ we deduce from (4) and (5) that for every m

$$\overline{\bigcup_n p_l(L_{mn}\alpha) + \mathcal{O}} = A \tag{8}$$

in case (a) and

$$\overline{\bigcup_n p_l(L_{mn}\alpha) + \mathcal{O}} = A_F\alpha + \mathcal{O} \tag{9}$$

in case (b). On the other hand, it follows from the definitions of H_m and y_n that for every $\delta > 0$ there exists c_δ such that if $\min\{m,n\} > c_\delta$ then $|x^{-2}\beta|_l < \delta$ and $||x|_j - 1| < \delta$ for all $x \in y_n H_m$. Now it follows from (6), (7), (8) and (9) that $U^+(A)\pi(e) \subset \overline{D_I\pi(h)}$ in case (a) and $U^+(A_F\alpha)\pi(e) \subset \overline{D_{I,\mathbb{R}}\pi(h)}$ in case (b).

Step 4. Let B_1^+ and $B_{1,\mathbb{R}}^+$ be the upper triangular subgroup of G_1 and $G_{1,\mathbb{R}}$, respectively. In view of *Step 3*, $B_1^+\pi(e) \subset \overline{D_I\pi(h)}$ in case (a) and $d_\alpha B_{1,\mathbb{R}}^+ d_\alpha^{-1}\pi(e) \subset \overline{D_{I,\mathbb{R}}\pi(h)}$ in case (b). Note that B_1^+ and $d_\alpha B_{1,\mathbb{R}}^+ d_\alpha^{-1}$ are epimorphic subgroups of G_1 and $d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}$, respectively. It follows from [Sh-W, Theorem 1] that $\overline{B_1^+\pi(e)} = \overline{G_1\pi(e)}$ and $\overline{d_\alpha B_{1,\mathbb{R}}^+ d_\alpha^{-1}\pi(e)} = \overline{d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}\pi(e)}$. Suppose we are in case (b). It is easy to see that $d_\alpha^{-1}\Gamma d_\alpha$ contains a congruence subgroup of Γ. Therefore $d_\alpha^{-1}\Gamma d_\alpha$ and Γ are commensurable and since $G_\mathbb{R}\pi(e)$ is closed $d_\alpha G_\mathbb{R} d_\alpha^{-1}\pi(e)$ is too. Using, for example, Borel's density theorem [R] one sees that $\overline{G_1\pi(e)} = G/\Gamma$ and $\overline{d_\alpha G_{1,\mathbb{R}} d_\alpha^{-1}\pi(e)} = d_\alpha G_\mathbb{R} d_\alpha^{-1}\pi(e)$. Therefore $\overline{D_I\pi(h)} = G/\Gamma$ in case (a) and $\overline{D_{I,\mathbb{R}}\pi(h)} \supset d_\alpha G_\mathbb{R} d_\alpha^{-1}\pi(e)$ in case (b). □

5.2. **Proofs of Theorem 1.5 and Corollary 1.7.** It is enough to prove Theorem 1.5 for $I = \{1,2,3\}$. We may (and will) assume that $g_i \in G_{i,K},\ i \in I$. By the theorem hypothesis either $g_1g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$ or $g_2g_3^{-1} \notin \mathcal{N}_{G_K}(D_K)$ (see Proposition 2.2). Suppose that $g_1g_2^{-1} \notin \mathcal{N}_{G_K}(D_K)$. In view of Proposition 3.5 there exists an element $\pi(g') \in \overline{D_I\pi(g)}$, $g' = (g_1', \cdots, g_r')$, such that $g_i' \in G_K$ if $1 \le i \le 3$, $g_1'g_2'^{-1} \in \mathcal{N}_{G_K}(D_K)$, $g_1'g_3'^{-1} \notin \mathcal{N}_{G_K}(D_K)$ and $g_i' = g_i$ if $i > 3$. Clearly, if $n \in D_I$ and $k \in G_K$ then $D_I\pi(g')$ is dense if and only if $D_I\pi(ng'k)$ is dense (see Lemma 3.1(c)). Therefore we may assume without loss of generality that $\overline{D_I\pi(g)}$ contains an element $\pi(h)$ where h is as in the formulation of Proposition 5.1. Now Theorem 1.5 follows from Proposition 5.1(a).

Let us prove Corollary 1.7. By Moore's ergodicity theorem [Z], $D_{I,\mathbb{R}}$ is ergodic on G/Γ. Therefore there exists an $y \in G/\Gamma$ such that $D_{I,\mathbb{R}}y$ is dense in G/Γ. By Theorem 1.5, $\overline{D_I\pi(g)} = G/\Gamma$. Therefore there exists a compact $M \subset D_I$ such that $M\overline{D_{I,\mathbb{R}}\pi(g)} = G/\Gamma$. Let $y = mz$, where $m \in M$ and $z \in \overline{D_{I,\mathbb{R}}\pi(g)}$. Then

$$\overline{D_{I,\mathbb{R}}\pi(g)} \supset m^{-1}\overline{D_{I,\mathbb{R}}y} = G/\Gamma$$

which completes the proof. □

5.3. **Proof of Theorem 1.8.** Recall that $I = \{1, \cdots, l\}$, $l \ge 3$. Choose $g = (e, \cdots, \underbrace{u_l^+(\alpha)u_l^-(\beta)}_{l}, \cdots, e)$ where $\alpha \in K \setminus F$, and $\beta \in F^*$.

We will prove that $x = \pi(g)$ is the point we need. First, remark that $u_l^+(\alpha)u_l^-(\beta) = tu_l^-(\beta_1)u_l^+(\alpha_1)$ where $t \in D_{l,K}$, $\beta_1 \in K$ and $\alpha_1 = \frac{\alpha}{1+\alpha\beta}$. Hence $\alpha_1 \in K \setminus F$. Let $d_{\alpha_1} \in D$ be such that $d_{\alpha_1}^2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}$. Applying twice Proposition 5.1(b) we obtain that

$$\overline{D_{I,\mathbb{R}}x} \supset G_{\mathbb{R}}\pi(e) \bigcup d_{\alpha_1} G_{\mathbb{R}} d_{\alpha_1}^{-1}\pi(e). \tag{10}$$

Note that the orbits $G_{\mathbb{R}}\pi(e)$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e)$ are closed and proper.

Since $G_{\mathbb{R}}\pi(e) \supset U^-(A_F)\pi(e)$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e) \supset U^+(A_F\alpha_1)\pi(e)$ we have that

$$\overline{D_{I,\mathbb{R}}x} \subset \{u_l^+(\mu\alpha)G_{\mathbb{R}}\pi(e) : 0 \leq \mu \leq 1\} \bigcup \{tu_l^-(\nu\beta_1)d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}\pi(e) : 0 \leq \nu \leq 1\},$$

where μ and $\nu \in F_l$. This implies (i).

Let us prove (ii). Using Proposition 2.4 we can choose a sequence $\xi_i \in \mathcal{O}_F^*$ such that for every $j \geq l$ the projection of ξ_i into F_j converges to some $x_j \in F_j^*$ and x_l is not an algebraic number. Put

$$y = (e, \cdots, u_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta), d_{l+1}(x_{l+1}^{-1}), \cdots, d_r(x_r^{-1}))\pi(e).$$

Then

$$y = \lim_i d_I(\xi_i)x \in \overline{D_{I,\mathbb{R}}x}.$$

Let us show that $y \notin D_{I,\mathbb{R}}x$. Otherwise, there exist elements $d \in D_l$ and $m \in G_{l,K}$ such that $du_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta) = u_l^+(\alpha)u_l^-(\beta)m$. Since $u_l^+(\alpha)u_l^-(\beta)m \in G_{l,K}$ the lower right coefficient of $du_l^+(x_l^2\alpha)u_l^-(x_l^{-2}\beta)$ belongs to K. This implies that $d \in D_{l,K}$ and that $x_l^2\alpha \in K$ which contradicts our choice of x_l, proving the claim.

Let H be a subgroup of G such that $H \supset D_{I,\mathbb{R}}$ and Hy be closed. It is easy to see that

$$x = \lim_i d_I(\xi_i^{-1})y.$$

In view of (10), H contains both $G_{\mathbb{R}}$ and $d_{\alpha_1}G_{\mathbb{R}}d_{\alpha_1}^{-1}$. Since $\alpha_1 \in K \setminus F$, we obtain $A = A_F + A_F\alpha_1$ and $A = A_F + A_F\alpha_1^{-1}$. Therefore, $H \supset U^+(A) \cup U^-(A)$. Hence $H = G$ which proves (ii).

In order to prove (iii), suppose on the contrary that

$$\overline{D_{I,\mathbb{R}}x} \setminus D_{I,\mathbb{R}}x \subset \bigcup_i H_i x_i$$

where $\{H_i\}$ is a countable family of proper subgroups of G and $H_i x_i$ are closed orbits. Then each H_i is closed. Let y be as in the formulation of (ii). It follows from the Baire category theorem that there exists H_j such that $D_{I,\mathbb{R}} \subset H_j$ and $y \in H_j x_j$. But the latter contradicts (ii). $\square$

5.4. **Proof of Corollary 1.9.** The fact that $\mathcal{N}_G(D_I)G_K$ is not equal to $\bigcap_{i\in I}\big(\mathcal{N}_G(D_i)G_K\big)$ is easy to prove. In view of Theorem 2.1(b) the orbit $D_I\pi(g)$ is locally divergent if and only if $g \in \bigcap_{i\in I}(\mathcal{N}_G(D_i)G_K)$ and in view of Proposition 2.2 if $g \in \mathcal{N}_G(D_I)G_K$ then $\overline{D_I\pi(g)}$ is an orbit of a torus. Suppose that $g \in \bigcap_{i\in I}\big(\mathcal{N}_G(D_i)G_K\big) \setminus \mathcal{N}_G(D_I)G_K$. Let $g = (g_1, \cdots, g_r)$. There exist i and $j \in I$, $i \neq j$, such that $g_i g_j^{-1}$ does not normalise the diagonal subgroup of $\mathrm{SL}(2)$.

We have seen in §4.1 when $\#I = 2$ and in §5.2 and §5.3 when $\#I > 2$ that in this case $\overline{D_I\pi(g)}$ is not an orbit of a torus. $\square$

6. A Number-theoretic Application

In this section we prove Theorem 1.10. We use the notation preceding the formulation of the theorem.

We identify the elements from G/Γ with the lattices in A^2 obtained via the injective map $g\Gamma \mapsto g\mathcal{O}^2$. This map is continuous and proper with respect to the quotient topology on G/Γ and the topology of Chabauty on the space of lattices in A^2.

The group G_K is acting on $K[X,Y]$ by the law

$$(\sigma p)(X,Y) = p(\sigma^{-1}(X,Y)), \forall \sigma \in G_K,\ \forall p \in K[X,Y],$$

where $\sigma^{-1}(X,Y) = (m_{11}X+m_{12}Y, m_{21}X+m_{22}Y)$, $\sigma^{-1} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. By the theorem hypothesis $f_i(X,Y) = l_{i,1}(X,Y)\cdot l_{i,2}(X,Y)$ where $l_{i,1}$ and $l_{i,2} \in K[X,Y]$ are linearly independent over K linear forms. There exist $g_i \in G_{i,K}$ such that $f_i(X,Y) = \alpha_i(g_i^{-1}f_0)(X,Y)$ where $\alpha_i \in K^*$ and f_0 is the form $X\cdot Y$. We may (and will) suppose that $\alpha_i = 1$ for all i. Since the forms $f_i,\ 1 \leq i \leq r$ are not proportional, $g = (g_1, \cdots, g_r)$ does not belong to $\mathcal{N}_G(D)G_K$. Therefore $D\pi(g)$ is a locally divergent non-closed orbit (Theorem 2.1(b)).

Let $r > 2$. Fix $a = (a_1, \cdots, a_r) \in A$ and choose $h \in G$ such that $he_1 = (a,1)$ where e_1 is the first vector of the canonical basis of the free A-module A^2. According to Theorem 1.5, $D\pi(g)$ is a dense orbit. Therefore there exist $d_n \in D$ and $\gamma_n \in \Gamma$ such that $\lim_n d_n g\gamma_n = h$. Put $z_n = \gamma_n e_1$. Then $z_n \in \mathcal{O}^2$ and

$$\lim_n f(z_n) = \lim_n f_0(d_n g\gamma_n e_1) = f_0(\lim_n (d_n g\gamma_n(e_1))) = f_0(a,1) = a,$$

which proves the part (a) of the theorem.

Let $r = 2$. We will prove the inclusion

$$\overline{f(\mathcal{O}^2)} \subset \bigcup_{j=1}^{s}\phi^{(j)}(\mathcal{O}^2)\bigcup(K_1' \times \{0\})\bigcup(\{0\}\times K_2')\bigcup f(\mathcal{O}^2), \tag{11}$$

where $\phi^{(j)}$, K_1' and K_2' are as in the formulation of the theorem. Let $a = (a_1, a_2) \in \overline{f(\mathcal{O}^2)} \setminus f(\mathcal{O}^2)$. There exists a sequence $z_n = (\alpha_n, \beta_n)$ in $\mathcal{O}^2$ such that $a = \lim_n f(z_n)$ and $f(z_n) \neq 0$ for all n. Let $a_1 \neq 0$. (The case $a_2 \neq 0$ is analogous.) If $f_2(z_n) = 0$ for infinitely many n then it is easy to see that $a \in K_1' \times \{0\}$. From now on we suppose that $f_2(z_n) \neq 0$ for all n.

Let $g = (g_1, g_2) \in G$ be such that $g_i(X,Y) = (l_{i1}(X,Y), l_{i2}(X,Y))$, $i \in \{1,2\}$. We choose sequences $s_n \in K_1^*$ and $t_n \in K_2^*$ such that

$$(12)\left\{\begin{array}{lcl} \lim_n s_n l_{11}(z_n) & = & a_{11} \\ \lim_n s_n^{-1} l_{12}(z_n) & = & a_{12} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \lim_n t_n l_{21}(z_n) & = & a_{21} \\ \lim_n t_n^{-1} l_{22}(z_n) & = & a_{22} \end{array}\right.$$

where $a_{11}, a_{12} \in K_1$, $a_{21}, a_{22} \in K_2$, $a_1 = a_{11} \cdot a_{12}$ and $a_2 = a_{21} \cdot a_{22}$.

If $a_2 = 0$ we choose t_n in such a way that

$$a_{21} = a_{22} = 0. \tag{13}$$

We have

$$\lim_n d(s_n, t_n) g(z_n) = (\mathbf{a_1}, \mathbf{a_2}) \tag{14}$$

where $\mathbf{a_1} = (a_{11}, a_{12}) \in K_1^2$ and $\mathbf{a_2} = (a_{21}, a_{22}) \in K_2^2$.

Shifting g from the left by an element from $\mathcal{N}_{G_K}(D_K)$ if necessary, we reduce the proof to the case when $|s_n|_1 \to \infty$ and $|t_n|_2 \leq 1$. There exist μ and $\nu \in K$ such that

$$l_{22} = \mu l_{11} + \nu l_{12}.$$

We have

$$\begin{aligned} 0 < |\mathrm{N}_{K/\mathbb{Q}}(l_{22}(z_n))| &= |l_{22}(z_n)|_1 \cdot |l_{22}(z_n)|_2 = \\ &= |s_n|_1 \cdot |t_n|_2 \cdot |\mu s_n^{-1} l_{11}(z_n) + \nu s_n^{-1} l_{12}(z_n)|_1 \cdot |t_n^{-1} l_{22}(z_n)|_2. \end{aligned}$$

Since $\{\mathrm{N}_{K/\mathbb{Q}}(l_{22}(z_n))\}$ is a discrete subset of $\mathbb{R}$ which does not contain 0, in view of (12), we obtain that

$$\liminf_n |s_n|_1 \cdot |t_n|_2 > 0 \tag{15}$$

and that

$$|a_{22}|_2 = \lim_n |t_n^{-1} l_{22}(z_n)|_2 \neq 0.$$

The latter contradicts (13). Hence $a_2 \neq 0$.

Let us prove that

$$g_1 g_2^{-1} \in B_K^- B_K^+. \tag{16}$$

First we need to show that

$$\limsup_n |s_n|_1 \cdot |t_n|_2 < \infty. \tag{17}$$

There exist μ' and $\nu' \in K$ such that

$$l_{11} = \mu' l_{21} + \nu' l_{22}.$$

Then

$$\begin{aligned} 0 < |\mathrm{N}_{K/\mathbb{Q}}(l_{11}(z_n))| &= |l_{11}(z_n)|_1 \cdot |l_{11}(z_n)|_2 = \\ &= |s_n|_1^{-1} \cdot |t_n|_2^{-1} \cdot |s_n l_{11}(z_n)|_1 \cdot |\mu' t_n l_{21}(z_n) + \nu' t_n l_{22}(z_n)|_2. \end{aligned}$$

Now (17) follows from the inequality $|t_n|_2 \leq 1$ and (12).

Suppose on the contrary that $g_1 g_2^{-1} \notin B_K^- B_K^+$. Therefore $g_1 g_2^{-1} \in \omega B_K^+$. Shifting g from the left by a suitable element from D_K we reduce the proof to the case when $g_1 g_2^{-1} = \omega u$, where $u = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. In view of (15), (17), Lemma 2.3 and Lemma 3.1 we can find a sequence $\xi_n \in \mathcal{O}^*$ and a converging to $a \in A^*$ sequence $a_n \in A^*$ such that $(s_n, t_n) = \xi_n a_n$ and $d(\xi_n) g_2 \mathcal{O}^2 = g_2 \mathcal{O}^2$. Using (14) we see that $d(\xi_n) g(z_n)$ converges to some $(\mathbf{b_1}, \mathbf{b_2}) \in A^2$ where $\mathbf{b_1} = (b_{11}, b_{12}) \in K_1^2$ and $\mathbf{b_2} = (b_{21}, b_{22}) \in K_2^2$. (Recall that A^2 is identify with $K_1^2 \times K_2^2$.) An easy computation shows that

$$d(\xi_n) g(z_n) = (h_n, e)\mathbf{w_n}$$

where $h_n = \begin{pmatrix} 0 & \xi_n^2 \\ -\xi_n^{-2} & -\alpha \end{pmatrix}$ and $\mathbf{w_n} = d(\xi_n) g_2(z_n) = (\beta_n, \gamma_n) \in g_2 \mathcal{O}^2$. So, $\big((\xi_n^2 \gamma_n, -\xi_n^{-2}\beta_n - \alpha\gamma_n), (\beta_n, \gamma_n)\big) \to (\mathbf{b_1}, \mathbf{b_2})$ which implies that $(\xi_n^2 \gamma_n, \gamma_n)$ converges to (b_{11}, b_{22}) in A. But

$$|\xi_n^2 \gamma_n|_1 \cdot |\gamma_n|_2 = |\xi_n^2|_1 \cdot |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)|.$$

Hence

$$\lim_n |\xi_n^2|_1 \cdot |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)| = |b_{11}|_1 \cdot |b_{22}|_2$$

which is a contradiction because $|\xi_n^2|_1 \to \infty$ and $\liminf_n |\mathrm{N}_{K/\mathbb{Q}}(\gamma_n)| > 0$. This completes the prove of (16).

In view of Proposition 3.3(b), there exists a subsequence of $d(s_n, t_n)\pi(g)$ converging to an element from $\bigcup_{j=1}^{s} D\pi(h_j)$, $2 \leq s \leq 4$ where $h_j \in \mathcal{N}_{G_K}(D_K)$ (see Corollary 1.3). So, there exists $d \in D$ such that $(\mathbf{a}_1, \mathbf{a}_2) \in dh_j \mathcal{O}^2$, $1 \leq j \leq s$. Hence $a \in \bigcup_{j=1}^{s} \phi^{(j)}(\mathcal{O}^2)$ where $\phi^{(j)} = h_j^{-1} f_0$. It is clear that the quadratic forms $\phi^{(j)}$ are 2×2 nonproportional. This completes the proof of (11).

The inclusion inverse to (11) is easy to prove. Let $c = \phi^{(j)}(z)$ where $z \in \mathcal{O}^2$. We have $h_j = \lim_n t_n g \sigma_n$ for some $t_n \in D$ and $\sigma_n \in \Gamma$.

Therefore

$$\phi^{(j)}(z) = \lim_n f_0(t_n g\sigma_n(z)) = \lim_n f(\sigma_n(z)) \in \overline{f(\mathcal{O}^2)}.$$

It remains to prove that $(K'_1 \times \{0\}) \bigcup (\{0\} \times K'_2) \subset \overline{f(\mathcal{O}^2)}$. It is enough to prove that if $(x,y) \in K_1^2$ and $f_2(x,y) = 0$ then $(f_1(x,y), 0) \in \overline{f(\mathcal{O}^2)}$. Suppose that $l_{21}(x,y) = 0$. Since l_{11} and l_{12} are linear combinations of l_{21} and l_{22} we get that $f_1(x,y) = c \cdot l_{22}(x,y)^2$ where c is a constant. Note that the projection of the set $\{l_{22}(z) : z \in \mathcal{O}^2, l_{21}(z) = 0\}$ into K_1 is dense. Therefore $(f_1(x,y), 0) \in \overline{f(\mathcal{O}^2)}$. By similar reasons if $l_{22}(x,y) = 0$ then $f_1(x,y) = d \cdot l_{21}(x,y)^2 \in \overline{f(\mathcal{O}^2)}$, where d is a constant. Note that $K'_1 = c\{\alpha^2 : \alpha \in K_1\} \cup d\{\alpha^2 : \alpha \in K_1\}$ and that, since f_1 and f_2 are not proportional, c and d can not be simultaneously equal to zero. This readily implies that $K'_i = \mathbb{C}$ if $K_i = \mathbb{C}$, and that $K'_i = \mathbb{R}, \mathbb{R}_-$ or $\mathbb{R}_+$ if $K_i = \mathbb{R}$. The proof is complete. □

7. Concluding remarks

The elements h_i in the formulation of Theorem 1.1 can be explicitly described in terms of g. This becomes clear from the proof of this theorem in §4.1. Here we will give an example of an orbit $D_I\pi(g)$, $I = \{1,2\}$, such that the boundary of its closure consists of *four* different closed orbits.

In the next proposition we suppose that G/Γ is a Hilbert modular space of rank $r = 2$. Let K_v be the completion of K with respect to a non-archimedean valuation v of K. Since K is dense in K_v we may (and will) choose α and $\beta \in K$ such that $\alpha \cdot \beta \neq 0$, $\alpha \cdot \beta \neq 1$, $|\alpha|_v > 1$ and $|\beta|_v < 1$.

Proposition 7.1. *With the above notation and assumptions, let* $g = (g_1, g_2) \in G$ *where* $g_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ *and* $g_2 = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. *Then*

$$\overline{D\pi(g)} \setminus D\pi(g) = \bigcup_{i=1}^{4} D\pi(h_i),$$

where $D\pi(h_i)$ *are pairwise different, closed, non-compact orbits.*

Proof. Since the coefficients of the matrix $g_1 g_2^{-1}$ are different from 0, all pairs $(\sigma_1, \sigma_2) \in \{0,1\}^2$ are admissibles and, in view of (3), we need to prove that the closed orbits $D(\omega^{\sigma_1}, \omega^{\sigma_2})\pi(h_{\sigma_1,\sigma_2})$ are pairwise different. We have seen in the course of the proof of Theorem 1.1 that $D(\omega^{\sigma_1}, \omega^{\sigma_2})\pi(h_{\sigma_1,\sigma_2}) \neq D(\omega^{\sigma'_1}, \omega^{\sigma'_2})\pi(h_{\sigma'_1,\sigma'_2})$ if $(\sigma_1, \sigma_2) = (0,0)$ or $(1,1)$ and $(\sigma'_1, \sigma'_2) = (0,1)$ or $(1,0)$. It remains to show that $D\pi(h_{0,0}) \neq D\pi(\omega h_{1,1})$ and $D(\omega, 1)\pi(h_{1,0}) \neq D(1,\omega)\pi(h_{0,1})$.

Using (2) we see that $h_{0,0} = e$ and modulo multiplication from the left by an element from D_K, $\omega h_{1,1}$ is equal to $\begin{pmatrix} \frac{1}{1-\alpha\beta} & \frac{\beta}{1-\alpha\beta} \\ \alpha & 1 \end{pmatrix}$. Since $\alpha \notin \mathcal{O}$ we conclude that $D\pi(h_{0,0}) \neq D\pi(\omega h_{1,1})$.

Modulo multiplication from the left by an element from D_K, $h_{1,0}$ (respectively, $h_{0,1}$) is equal to $\begin{pmatrix} 1 & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix}$ (respectively, $\begin{pmatrix} 1 & 0 \\ \frac{1}{\beta} & 1 \end{pmatrix}$). If $D(\omega, 1)\pi(h_{1,0}) = D(1, \omega)\pi(h_{0,1})$ then

$$\frac{\xi^2\beta + \alpha}{\alpha\beta} \in \mathcal{O}$$

for some $\xi \in \mathcal{O}^*$. This leads to contradiction because, in view of the choice of α and β,

$$\frac{|\xi^2\beta + \alpha|_v}{|\alpha\beta|_v} = \frac{1}{|\beta|_v} > 1.$$

Therefore the boundary of $D_I\pi(g)$ consists of four pairwise different closed orbits. □

Remark that most of the results of this paper remain valid with very small changes in the S-adic setting, that is, when G is a product of $\mathrm{SL}(2, K_v)$, where K_v is the completion of a number field K with respect to a place v belonging to a finite set S of places of K containing the archimedean ones. For instance, the proofs of the analogs of Theorems 1.1 and 1.10(b) remain valid in this context with virtually no changes. When K is not a CM-field, the analog of Theorem 1.5 remains true with very small modifications if $K = \mathbb{Q}$ or if I contains an archimedean place. For instance, Theorem 1.5 remains true for action of maximal tori (that is, when $D = D_I$). The analog of Theorem 1.5 in the general case (for arbitrary K and I) is more delicate and will be treated later. Also, one can find tori orbits with non-homogeneous closures for many spaces G/Γ with $G \neq \mathrm{SL}_n$. This will be treated elsewhere too.

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